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On the second-order solution to the Sears problem for compressible flow

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Significant simplifications and minor corrections are made to a previous second-order solution of Graham & Kullar for the lift on a flat-plate airfoil encountering a sinusoidal gust in compressible flow. The related cases of a skewed gust in incompressible flow, a parallel gust in compressible flow and the generalized case of a skewed gust in compressible flow are considered. In addition to the simplifications, the solutions are combined into a composite solution that is more accurate than the solutions from which it is composed, making it useful for numerical calculations.

1. Introduction

Graham & Kullar (1977) presented a second-order solution to the gust response of an airfoil in compressible flow. A closed-form solution to this problem is well known for the case of a parallel gust incident on a flat-plate airfoil in incompressible flow, the classic treatment being that of von Kármán & Sears (1938). When compressibility is added to the problem, the solution becomes significantly more complex, and no finite closed-form solution is available. For this problem one must resort to numerical solutions or approximate methods such as expansion in a small parameter. A first-order solution was given by Osborne (1973) and Amiet (1974). In order to clarify the problem further, a second-order solution was given by Graham & Kullar (1977). Little attention has been given to the problem since then.

Recently a review of Graham & Kullar's solution was undertaken with the purpose of incorporating it into an approximation scheme for the calculation of lift as illustrated by Amiet (1975). The derivation of the Graham & Kullar solution is quite involved and the author is grateful to Professor Graham for providing a copy of his report (Graham 1978) giving further details on the solution, since this proved invaluable in the analysis. In the review process significant algebraic simplifications were discovered. These do not at all change the theoretical impact of the solution, but they clarify it and make it more useful for numerical calculation. The original solution is an involved expression containing Bessel functions, and it was not at all evident *a priori* that any simplification would be possible.

Graham & Kullar give two different approximations for each case they considered. Thus, for the case of a skewed vertical gust in incompressible flow the first approximation is to assume a small spanwise wavenumber, k_y, while the chordwise wavenumber, k_x, is arbitrary. This approximation becomes invalid if the limit $k_x \to 0$ is taken for fixed k_y. Thus, a second approximation is introduced taking $k_x = O(k_y)$ and assuming that both become small. A similar procedure was used for the problems of a parallel gust and a skewed gust in compressible flow.

In this paper a method is found to combine these two solutions for each case to give a composite solution which reduces under the appropriate conditions to each of the two approximations of which it is composed. This does not affect any of the theoretical conclusions drawn by Graham & Kullar, but rather it adapts the solution for the calculation of numerical results by eliminating the need to switch from one solution to the other. If the result for a parallel gust in compressible flow is used together with an approximate high-frequency solution it is found that the two solutions cover all k_x and M values with an accuracy of approximately 1 % for the lift amplitude when compared with a numerical exact solution. For the skewed gust case the results appear to be somewhat less accurate, but still satisfactory. In addition, because the appropriate solutions combine so neatly to form the composite solution, it is thought that the process leads to a further understanding of the general gust solution. Since the composite solution retains the mathematical accuracy of both the individual solutions that go to make it up, it is possible that a further examination of the solution procedure would reveal an expansion procedure that would arrive at the composite solution directly.

2. The problem formulation

First the solution of Graham & Kullar (1977) will be presented; the reader is referred to that paper for details of their solution procedure. Just a brief review of the integral equation to be solved will be presented here before proceeding to the simplifications.

The vertical gust, imbedded in the main stream, is defined as

$$w(x,y,t) = w_0\, U\, \mathrm{e}^{\mathrm{i}(\omega t - k_x x - k_y y)}, \tag{1}$$

where x and k_x represent the chordwise length and wavenumber and y and k_y represent the spanwise variables; U is the free-stream velocity and w_0 is the gust amplitude normalized by U. A sketch of the gust and airfoil is shown in figure 1. Since the gust is imbedded in the main stream, the circular frequency ω is equal to Uk_x, and so k_x is a reduced frequency as well as a wavenumber.

The integral equation to be solved is given by Graham & Kullar as

$$\int_{-1}^{1} \nu F(\xi)\, K_1[\nu(\xi - x)]\, \mathrm{d}\xi = -\pi w_0\, \mathrm{e}^{-\mathrm{i}\lambda x} + \nu^2 K_0[\nu(1-x)] \int_{-1}^{1}\int_{-1}^{\xi} f(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi$$
$$-\mathrm{i}\lambda\left\{(\nu^2/\lambda^2)\, K_0[\nu(1-x)] - (1+\nu^2/\lambda^2)\int_0^{\infty} \nu\, \mathrm{e}^{-\mathrm{i}\lambda\xi} K_1[\nu(1+\xi-x)]\,\mathrm{d}\xi\right\}\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi, \tag{2}$$

where $\quad \nu^2 = \nu_2^2 - \nu_1^2, \quad \nu_1 = M\lambda, \quad \nu_2 = k_y/\beta, \quad \lambda = k_x/\beta^2, \quad \beta = (1-M^2)^{\frac{1}{2}}. \qquad (3)$

The functions K_0 and K_1 are modified Bessel functions of the second kind; in addition in both the derivation of (2) and its solution the following convention is used for negative arguments for the modified Bessel functions:

$$K_n(z\,\mathrm{e}^{\mathrm{i}\pi m}) = \mathrm{e}^{-\mathrm{i}\pi m n}\, K_n(z). \tag{4}$$

Also, the integrals containing K_1 are Cauchy principal values. The functions $f(x)$ and $F(x)$ are related by

$$F(x) = f(x) - \nu^2 \int_{-1}^{x}\int_{-1}^{\xi} f(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{5}$$

Having determined $f(x)$, the airfoil loading is given by

$$C_{\Delta p}(x) = (2/\beta)\, \mathrm{e}^{\mathrm{i}k_x M^2 x/\beta^2}\left[f(x) + \mathrm{i}\lambda \int_{-1}^{x} f(\xi)\,\mathrm{d}\xi\right]. \tag{6}$$

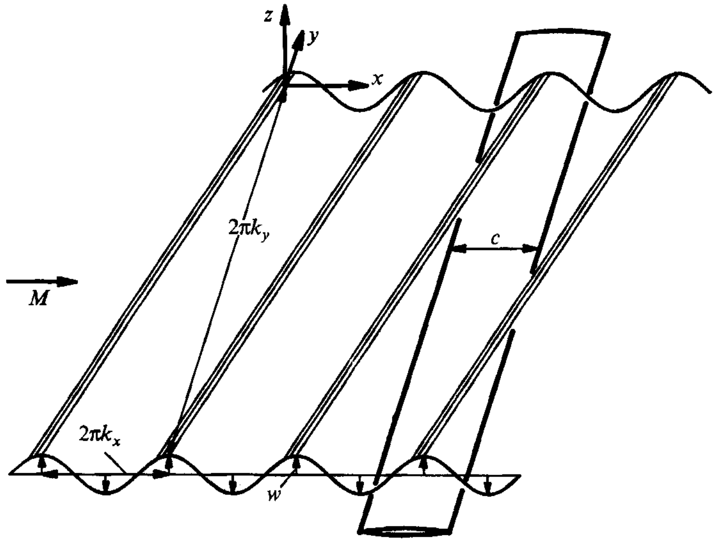

FIGURE 1. Sketch of an oblique sinusoidal gust incident on an airfoil.

These are exactly the relations given by Graham & Kullar (except for the sign of iλ in (6) compared to equation (2) of Graham & Kullar, which appears to be a misprint). The integral equation is to be solved for the variable F. This equation contains the variables k_x, k_y and the Mach number M in only the two parameters ν and λ. Thus, having solved the integral equation for the zero Mach number case, the same solution for F will apply for all M for which ν and λ remain unchanged. However, the expression for the surface loading and the lift must be calculated from F; since (6) contains M explicitly, the expression for lift will change if M becomes non-zero even if ν and λ remain unchanged. Thus, to find the lift for the general case of a skewed gust in compressible flow, the three variables ν_1, ν_2 and λ must be specified; from (3) this is the same as specifying k_x, k_y and M.

For a clearer understanding of this formulation, it is worthwhile to consider its origin. Graham (1970*b*) derived (2) beginning with a form of the Posio integral equation,

$$\int_{-1}^{1} \Delta p(\xi)\, K(x-\xi)\, \mathrm{d}\xi = 4\pi w(x), \tag{7}$$

given by Watkins, Runyon & Woolston (1955, equations (1) and (B 18)). Δp is the pressure jump across the surface, w is the upwash and K is a kernel function, given by equation (B 18) of Watkins *et al.* or Graham (1970*b*, equation (23)). The transformation to (2) above is then made by replacing Δp using (6) and manipulating the result. Although the Posio integral equation is for a non-skewed gust (the two-dimensional case), Graham (1970*b*) has given a similarity relation between the two-dimensional and three-dimensional compressible gust cases which is used to extend the two-dimensional Posio integral equation to the three-dimensional result in (2).

As a further clarification of the solution, it is useful to have a better understanding of the variable $f(x)$. Consider (6) for the case of incompressible flow, $M = 0$. This equation can be compared with a standard result of airfoil theory given, for example by equation (6) of Amiet (1990) which relates the bound vorticity, $\gamma(x)$, of the airfoil to the airfoil loading $\Delta p(x)$. For a sinusoidal time dependence, (6) above is seen to correspond with (6) of Amiet if $f(x) = \gamma(x)/(2U)$. In fact, the solution for the incompressible case is often formulated directly using the bound vorticity as, for example, von Kármán & Sears (1938) wherein equation (16) gives the overall lift in terms of the bound vorticity.

3. The solution and its simplification

3.1. *Skewed gust in incompressible flow*

Consider the case of a skewed gust in incompressible flow. For the limit of zero spanwise wavenumber, ν_2, the solution is exactly the Sears function, $S(\lambda)$. Thus, assuming a small parameter ν_2, Graham & Kullar express the functions $F(x)$ and $f(x)$ and the lift coefficient, C_L, in powers of ν_2. Introducing these expansions into (2) and equating powers of ν_2 leads to a sequence of integral equations. It is possible to solve these integral equations, the result up to second order being

$$\left[\frac{C_L(k_x, k_y)}{2\pi w_0}\right]_{\nu_2\to 0} = S(\lambda) + \left[\frac{1}{4} + \frac{C(\lambda)}{2\mathrm{i}\lambda}\right] S(\lambda)\,\nu_2^2 \ln \nu_2 + [A(\lambda)\,S(\lambda) + B(\lambda)]\,\nu_2^2 + O(\nu_2^3), \quad (8)$$

where for the $M = 0$ case $\lambda = k_x$, $\nu_1 = 0$ and $\nu_2 = k_y$. The lift coefficient is defined as the lift/unit span normalized by $\frac{1}{2}c\rho U^2$ where c, U and ρ are the chord, the free-stream velocity and the density. Because several small-parameter expansions are taken in this paper, in order to retain clarity the expansions are shown explicitly; thus, $C_L(k_x, k_y)$ represents the exact lift coefficient for the incompressible skewed gust case and $[C_L(k_x, k_y)]_{\nu_2\to 0}$ represents the expansion of this for small ν_2. $S(\lambda)$ and $C(\lambda)$ represent the Sears function and the Theodorsen function respectively, given by

$$S(\lambda) = \frac{2}{\pi\lambda}\frac{1}{[H_0^{(2)}(\lambda) - \mathrm{i}H_1^{(2)}(\lambda)]} \quad (9)$$

and

$$C(\lambda) = \frac{H_1^{(2)}(\lambda)}{H_1^{(2)}(\lambda) + \mathrm{i}H_0^{(2)}(\lambda)}, \quad (10)$$

where $H_n^{(2)}$ are the Hankel functions of the second kind of order n. The functions $A(\lambda)$ and $B(\lambda)$, in the form given by Graham & Kullar and with which this author agrees, are

$$A(\lambda) \equiv \tfrac{1}{4}(\gamma - 2\ln 2) + (\gamma - 2\ln 2 - \tfrac{3}{2})\frac{C(\lambda)}{2\mathrm{i}\lambda} \quad (11)$$

and

$$\begin{aligned} B(\lambda) \equiv{}& C(\lambda)\left(\frac{3}{8} - \frac{\mathrm{i}}{4\lambda}\right) J_0(\lambda) + \frac{\mathrm{i}}{4\lambda} J_0(\lambda) \\ & - C(\lambda)\left(\frac{3\mathrm{i}}{8} - \frac{1}{2\lambda} - \frac{\mathrm{i}}{2\lambda^2}\right) J_1(\lambda) - J_1(\lambda)\left(\frac{\mathrm{i}}{8} + \frac{\mathrm{i}}{2\lambda^2}\right) - \tfrac{1}{2}C(\lambda)\,[J_0(\lambda) - \mathrm{i}J_1(\lambda)] \\ & + \frac{J_0(\lambda) - \mathrm{i}J_1(\lambda)}{2\mathrm{i}\lambda[H_1^{(2)}(\lambda) + \mathrm{i}H_0^{(2)}(\lambda)]}\left\{\frac{\mathrm{i}}{2}H_0^{(2)}(\lambda)\,[1 - C(\lambda)] + H_1^{(2)}(\lambda)\left[\frac{1}{\mathrm{i}\lambda} + C(\lambda)\left(\frac{\mathrm{i}}{\lambda} + 1\right)\right]\right\}. \end{aligned} \quad (12)$$

$B(\lambda)$ is of sufficient complexity that one might hesitate to use it for calculations.

However, it can be greatly simplified. With a significant amount of manipulation the Sears function $S(\lambda)$ can be factored out and $B(\lambda)$ can be written exactly as

$$B(\lambda) = -\tfrac{1}{8}S(\lambda)\left[\frac{4}{\lambda^2}-\frac{2\mathrm{i}}{\lambda}+1+\frac{2\mathrm{i}}{\lambda}\left(\frac{2\mathrm{i}}{\lambda}+3\right)C(\lambda)\right]. \tag{13}$$

Using this result, the function $A(\lambda)\,S(\lambda)+B(\lambda)$ in (8) becomes

$$\frac{A(\lambda)\,S(\lambda)+B(\lambda)}{S(\lambda)} = (\gamma-2\ln 2)\left[\frac{1}{4}+\frac{C(\lambda)}{2\mathrm{i}\lambda}\right]+\frac{C(\lambda)-1}{2\lambda^2}+\frac{\mathrm{i}}{4\lambda}-\frac{1}{8} \tag{14}$$

This represents a significant simplification over (11) and (12), and shows that the Sears function $S(\lambda)$ is an overall factor of the lift coefficient for the second-order solution. Because of the detailed nature of the algebraic derivation, (12) and (13) were checked by calculating each for specific λ value and affirming numerical equality.

As noted by Graham & Kullar, (8) does not hold for $\lambda = O(\nu_2)$. The term containing the factor $\nu_2^2 \ln \nu_2$ in (8) becomes infinite for $\lambda \to 0$. For this range another derivation was made with the assumption that $\lambda = \alpha\nu_2 \to 0$ with $\alpha = O(1)$. This leads to the result

$$\left[\frac{C_L(k_x k_y)}{2\pi w_0}\right]_{\lambda=\alpha\nu_2\to 0} = 1+\mathrm{i}\lambda\ln\nu_2+\mathrm{i}\lambda(\gamma-2\ln 2)-\eta\nu_2\,\delta+\eta^2\nu_2^2(\tfrac{1}{2}\gamma-\tfrac{1}{4}-\ln 2+\delta^2+\tfrac{1}{2}\ln\nu_2)$$
$$+\mathrm{i}\nu_2\,\lambda\eta\delta(\tfrac{1}{2}-2\gamma+4\ln 2-2\ln\nu_2)-\lambda^2(\gamma-2\ln 2+\ln\nu_2)^2+O(\nu_2^3), \tag{15}$$

with an argument of ν_2/λ assumed for η and δ, where

$$\eta(\nu_2/\lambda) = (1+\lambda^2/\nu_2^2)^{\frac{1}{2}}, \quad \delta(\nu_2/\lambda) = \tfrac{1}{2}\pi-\mathrm{i}\ln(\eta+\lambda/\nu_2). \tag{16}$$

Again this agrees exactly with Graham & Kullar's solution. As pointed out by Graham & Kullar, this reduces to the appropriate limits for both cases $k_x \to 0$ and $k_y \to 0$. Since

$$S(\lambda) = \frac{1+\tfrac{1}{2}\lambda^2(C_0-\tfrac{1}{2}+\ln\lambda)}{1-\mathrm{i}\lambda(C_0+\ln\lambda)}+O(\lambda^3), \tag{17}$$

where

$$C_0 \equiv \tfrac{1}{2}\mathrm{i}\pi+\gamma-\ln 2 \tag{18}$$

and γ is Euler's constant, one notes from (15) that C_L reduces to the small-λ expansion of $S(\lambda)$ for $k_y = 0$ as expected. For $k_x = 0$ one finds from (15) that

$$\left[\frac{C_L(0,k_y)}{2\pi w_0}\right]_{k_y\to 0} = 1-\tfrac{1}{2}\pi k_y+\tfrac{1}{2}k_y^2\ln k_y+(\tfrac{1}{2}\gamma-\tfrac{1}{4}-\ln 2+\tfrac{1}{4}\pi^2)\,k_y^2, \tag{19}$$

which Graham & Kullar note is the small-perturbation expansion for a sinusoidally twisted wing of infinite span in steady flow. An expansion for $C(\lambda)$ useful in the derivations is

$$C(\lambda) = [1-\mathrm{i}\lambda(C_0+\ln\lambda)]^{-1}+O(\lambda^3). \tag{20}$$

Written in this manner, there are no explicit $O(\lambda^2)$ terms.

3.2. *Parallel gust in compressible flow*

As noted above, the same solution of the integral equation (2) for $F(x)$ holds for this case by taking $\lambda = k_x/\beta^2$ and $\nu = \mathrm{i}k_x M/\beta^2 = \mathrm{i}\nu_1$. Thus, the results for the incompressible skewed gust given above represent the major part of the effort needed in the derivation of the following cases of a parallel gust in compressible flow and a skewed gust in compressible flow. The only additional change needed to the previous solution is to account for the exponential factor in (6). With this consideration, the solution corresponding to (8) for this case with $\nu_1 \to 0$ was found to be

$$\left[\frac{\beta C_L(\lambda, \nu_1)}{2\pi w_0 S(\lambda)}\right]_{\nu_1\to 0} = 1 - \left\{(C_0 - \ln 2 + \ln \nu_1)\left[\frac{1}{2} + \frac{C(\lambda)}{\mathrm{i}\lambda}\right] + \frac{C(\lambda)-1}{\lambda^2} + \frac{3\mathrm{i}}{2\lambda} - \frac{1}{4}\right\}\frac{\nu_1^2}{2} + O(\nu_1^3). \tag{21}$$

Corresponding to (15) the result for $\lambda = O(\nu_1) \to 0$ was found to be

$$\left[\frac{\beta C_L(\lambda, \nu_1)}{2\pi w_0}\right]_{\lambda=\alpha\nu_1\to 0} = \frac{1}{1 - \mathrm{i}\lambda[C_0 + \ln\lambda + f(M)]} - \tfrac{1}{2}\mathrm{i}\lambda M^2 + \tfrac{1}{2}\lambda^2\left[\tfrac{1}{2}M^2\beta^2 - M^2\ln\tfrac{1}{2}M - \tfrac{1}{2} + C_0 + \ln\lambda + f(M)\right] + O(\nu_1^3), \tag{22}$$

with

$$f(M) \equiv (1-\beta)\ln M + \beta\ln(1+\beta) - \ln 2. \tag{23}$$

The function $f(M)$ was first introduced by Miles (1951), and should not be confused with $f(x)$ in (2), (5) and (6); $f(x)$ is not used explicitly in the remainder of this paper. The first term on the right-hand side of (22) was written in the manner shown only to simplify the expression; if desired, it could be expanded for small λ. Again, these expressions are the same as those of Graham & Kullar, except that there are a few minor differences between (22) (after expanding the first term for small λ) and their corresponding result. Also, (14) has been used to simplify (21) compared to the result given by Graham & Kullar.

3.3. *Skewed gust in compressible flow*

Finally, for the general case of a skewed gust in compressible flow, the result corresponding to (8) is

$$\left[\frac{\beta C_L(\lambda, \nu, M)}{2\pi w_0 S(\lambda)}\right]_{\nu\to 0} = 1 + \nu^2\left\{(\gamma - 2\ln 2 + \ln\nu)\left[\frac{1}{4} + \frac{C(\lambda)}{2\mathrm{i}\lambda}\right] + \frac{C(\lambda)-1}{2\lambda^2} + \frac{\mathrm{i}}{4\lambda} - \frac{1}{8}\right\} - \frac{\mathrm{i}\nu_1^2}{2\lambda} + O(\nu^3), \tag{24}$$

where it was necessary to assume both $\nu_1 \to 0$ and $\nu_2 \to 0$. Thus, the case $\nu_1 = \nu_2 = O(1)$ is excluded even though it satisfies $\nu = 0$; this case will be considered later. Corresponding to (15), the result for $\lambda = O(\nu_1) = O(\nu_2) \to 0$ is

$$\begin{aligned}\left[\frac{\beta C_L(\lambda, \nu, M)}{2\pi w_0}\right]_{\lambda=\alpha\nu\to 0} = {} & 1 + \mathrm{i}\lambda\ln\nu + \mathrm{i}\lambda(\gamma - 2\ln 2) - \eta\nu\delta + \frac{\nu_1^2}{2\mathrm{i}\lambda} + \tfrac{1}{2}\nu_1^2(C_0 + \ln\lambda) \\ & + [\tfrac{1}{2}\eta^2\nu^2 - 2\lambda^2(\gamma - 2\ln 2) - 2\mathrm{i}\lambda\nu\eta\delta]\ln\nu - \frac{\nu_1^4}{4\lambda^2} - \lambda^2(\ln\nu)^2 \\ & + \eta^2\nu^2(\tfrac{1}{2}\gamma - \tfrac{1}{4} - \ln 2 + \delta^2) + \mathrm{i}\nu\lambda\eta\delta(\tfrac{1}{2} - 2\gamma + 4\ln 2) \\ & - \lambda^2(\gamma - 2\ln 2)^2 + O(\nu^3)\end{aligned} \tag{25}$$

with an argument of ν/λ assumed for η and δ where

$$\eta(\nu/\lambda) = (1 + \lambda^2/\nu^2)^{\frac{1}{2}}, \quad \delta(\nu/\lambda) = \tfrac{1}{2}\pi - \mathrm{i}\ln(\eta + \lambda/\nu). \tag{26}$$

Again, this is basically the result of Graham & Kullar (except for a sign difference in (24) and a few small differences in (25)), but with (14) introduced as a simplification. Equation (24) encompasses both (8) and (21), and (25) encompasses both (15) and (22).

4. Composite solutions

4.1. Skewed gust in incompressible flow

It is possible to fairly simply combine these three pairs of solutions to form three composite solutions by noting a fundamental relation between the solution pairs. First (8) and (15) will be combined. Using (17) to factor $S(\lambda)$ from (15) gives

$$\left[\frac{C_L(k_x,k_y)}{2\pi w_0 S(\lambda)}\right]_{\lambda=\alpha\nu_2\to 0} = 1+\mathrm{i}\lambda g+\tfrac{1}{2}(\eta^2\nu_2^2-2\lambda^2 g)g-\lambda^2(C_0+\ln\lambda)g$$
$$+\tfrac{1}{2}\nu_2^2(-\tfrac{1}{2}+C_0+\ln\lambda)-\mathrm{i}\nu_2\lambda\eta\delta\frac{\nu_2^2}{2\lambda^2}+O(\nu_2^3), \quad (27)$$

with an argument ν_2/λ assumed for g, η and δ and where

$$g(\nu_2/\lambda) = \ln(\nu_2/\lambda)-\tfrac{1}{2}\mathrm{i}\pi+(\mathrm{i}\nu_2/\lambda)\eta\delta-\ln 2. \quad (28)$$

The functions η and δ are given by (16).

Now the opposite expansions of (8) and (27) will be taken; that is, the $\lambda=\alpha\nu_2\to 0$ expansion of (8), with $\alpha = O(1)$, and the $\nu_2\to 0$ expansion of (27) will be made. Taking the $\nu_2\to 0$ expansion of (27) might at first thought seem pointless since the equation has already been expanded for small ν_2 through the expansion $\lambda=\alpha\nu_2\to 0$. However, this keeps ν_2 and λ in a fixed ratio. In contrast, when $\nu_2 \ll \lambda$, factors such as δ and η that remain in (27) become simplified in (29) below. Expanding (27) gives

$$\left\{\left[\frac{C_L(k_x,k_y)}{2\pi w_0 S(\lambda)}\right]_{\lambda=\alpha\nu_2\to 0}\right\}_{\nu_2\to 0} = 1+\tfrac{1}{2}\nu_2^2\frac{\mathrm{i}}{\lambda}(C_0+\ln\lambda+2\ln 2-\gamma-\ln\nu_2+\tfrac{1}{2})-\tfrac{1}{2}\nu_2^2(C_0+\ln\lambda)^2$$
$$-\tfrac{1}{8}\nu_2^2+\tfrac{1}{2}\nu_2^2(\gamma-2\ln 2+\ln\nu_2)(C_0+\ln\lambda+\tfrac{1}{2})+O(\nu_2^3). \quad (29)$$

This is found to be the same result as the expansion of (8) for $\lambda=\alpha\nu_2\to 0$ with $\alpha=O(1)$. That is,

$$\{[C_L(k_x k_y)/S(\lambda)]_{\lambda=\alpha\nu_2\to 0}\}_{\nu_2\to 0} = \{[C_L(k_x,k_y)/S(\lambda)]_{\nu_2\to 0}\}_{\lambda=\alpha\nu_2\to 0}. \quad (30)$$

This observation provides the means to easily derive a composite solution. Thus, define a general solution as the sum of (8) and (27) with (29) subtracted. That is,

$$\frac{[C_L(k_x,k_y)]_{composite}}{S(\lambda)} = \left[\frac{C_L(k_x,k_y)}{S(\lambda)}\right]_{\nu_2\to 0}+\left[\frac{C_L(k_x,k_y)}{S(\lambda)}\right]_{\lambda=\alpha\nu_2\to 0}-\left\{\left[\frac{C_L(k_x,k_y)}{S(\lambda)}\right]_{\lambda=\alpha\nu_2\to 0}\right\}_{\nu_2\to 0} \quad (31)$$

The composite result for the skewed gust in incompressible flow is thus

$$\frac{[C_L(k_x,k_y)]_{composite}}{2\pi w_0 S(\lambda)} = 1+\mathrm{i}\lambda g+\left\{(\gamma-2\ln 2+\ln\nu_2)\left[1+\frac{C(\lambda)}{\mathrm{i}\lambda}\right]+\frac{C(\lambda)-1}{\lambda^2}-\frac{1}{2}\right\}\frac{\nu_2^2}{2}$$
$$-\lambda^2 g^2-\lambda^2 g(C_0+\ln\lambda-\tfrac{1}{2})+\tfrac{1}{2}\nu_2^2\left(C_0+\ln\lambda+\frac{1}{\mathrm{i}\lambda}\right)(C_0+\ln\lambda-\gamma+2\ln 2-\ln\nu_2), \quad (32)$$

with an argument ν_2/λ assumed for g. Because of (30) this result is easily seen to reduce to the proper limits for $\nu_2\to 0$ and for $\lambda=\alpha\nu_2\to 0$. It also reduces to (19) for $\lambda\to 0$. This can be compared to the first-order solution of Amiet (1976*b*); the function $g(\nu_2/\lambda)$ here is the function g in (7) of that paper. There is a misprint of an extra bracket), in the first line of that equation. Also, $g(\nu_2/\lambda)$ is the function $f(M)$ in equation (8*b*) of Amiet (1974) if one substitutes $\mathrm{i}M$ for ν_2/λ.

By combining the two solutions in this manner, however, a new difficulty was introduced. Although the composite solution does give the proper limit for $\lambda \to 0$ with ν_2 fixed, it does not approach the point $\lambda = 0$ with the proper slope. The $\lambda \to 0$ limit of (15) gives the correct limit which is

$$\frac{C_L(\nu_2)}{2\pi w_0} = 1 + i\lambda(1-\pi\nu_2)(\gamma + \ln\nu_2 + 1 - 2\ln 2) + \tfrac{1}{4}i\lambda\pi\nu_2 - \tfrac{1}{2}\pi\nu_2$$
$$+ \tfrac{1}{2}\nu_2^2(\tfrac{1}{2}\pi^2 + \gamma - \tfrac{1}{2} - 2\ln 2 + \ln\nu_2)[1 + i\lambda(C_0 + \ln\lambda)] + O(\nu_2^3). \quad (33)$$

In the process of forming a composite solution this result was altered. A correction of the lift result for this difficulty is made in §6 below.

It is worth noting that the ability to combine the two solutions into a composite one in the manner performed here depends on (30). This type of relation is violated for even simple functions, although it appears to hold in the present case. A simple example for which it does not hold is

$$f(x,y) = \frac{x^2y^2}{x^2+y^2}. \quad (34)$$

The two expansion procedures above, $\nu_2 \to 0$ and $\lambda = O(\nu_2) \to 0$, could be restated using α^{-1} instead of ν_2 as the independent variable, based on the relation $\lambda = \alpha\nu_2$. The two expansions then become $\alpha^{-1} \to 0$ with λ fixed and $\lambda \to 0$ with α fixed. Thus, for illustration purposes, expanding (34) for $x \to 0$ and $y \to 0$ can be considered an equivalent procedure to the expansions above. Expanding (34) for $x \to 0$ gives $(1 - x^2/y^2)x^2$ and expanding it for $y \to 0$ gives $(1 - y^2/x^2)y^2$. Taking the opposite expansions, the dominant term for the $y \to 0$ expansions of the former result gives x^4/y^2 which is not equal to the dominant term for the $x \to 0$ expansion of the latter which is y^4/x^2.

4.2. *Parallel gust in compressible flow*

The analysis proceeds in the same manner as for the above case. Factoring $S(\lambda)$ out of (22) gives

$$\left[\frac{\beta C_L(\lambda,\nu_1)}{2\pi w_0 S(\lambda)}\right]_{\lambda=\alpha\nu_1\to 0} = 1 + i\lambda(f(M) - \tfrac{1}{2}M^2) - \lambda^2 f^2(M)$$
$$- \lambda^2 f(M)(C_0 + \ln\lambda - \tfrac{1}{2}) - \tfrac{1}{2}\lambda^2 M^2(C_0 + \ln\lambda - \tfrac{1}{2}\beta^2 + \ln\tfrac{1}{2}M) + O(\nu_1^3). \quad (35)$$

Expanding for $\nu_1 \to 0$ gives

$$\left\{\left[\frac{\beta C_L(\lambda,\nu_1)}{2\pi w_0 S(\lambda)}\right]_{\lambda=\alpha\nu_1\to 0}\right\}_{\nu_1\to 0} = 1 + i\lambda[(\ln\tfrac{1}{2}M - \tfrac{1}{2}) - 1]\tfrac{1}{2}M^2 - \lambda^2\tfrac{1}{2}M^2(C_0 + \ln\lambda)$$
$$- \lambda^2\tfrac{1}{2}M^2(\ln\tfrac{1}{2}M - \tfrac{1}{2})(C_0 + \ln\lambda + \tfrac{1}{2}) + O(\nu_1^3). \quad (36)$$

Combining (21), (35) and (36) as in (31) gives for the composite solution

$$\frac{\beta[C_L(\lambda,\nu_1)]_{composite}}{2\pi w_0 S(\lambda)} = 1 + i\lambda\left[f(M) - \tfrac{1}{2}M^2(\ln\tfrac{1}{2}M + 1)\right]$$
$$- \left\{(C_0 - \ln 2 + \ln\nu_1)\left[1 + \frac{C(\lambda)}{i\lambda}\right] + \frac{C(\lambda)-1}{\lambda^2} - \frac{1}{2}\right\}\frac{\nu_1^2}{2}$$
$$- \lambda^2[f(M)]^2 - \lambda^2 f(M)(C_0 + \ln\lambda - \tfrac{1}{2})$$
$$+ \tfrac{1}{2}\nu_1^2(\ln\tfrac{1}{2}M)(C_0 + \ln\lambda) - \nu_1^2\tfrac{1}{4}M^2. \quad (37)$$

This is better behaved than the corresponding equation (32) for the preceding case. For that case there was a difficulty when $\lambda \to 0$. Using the similarity, $\nu_2/\lambda \to \mathrm{i}M$, the corresponding limit for the present case is $M \to \mathrm{i}\infty$, which is not a case of interest. Thus, (37) is satisfactory as it is, and a correction term is not necessary, although the solution will be put in a preferable form in §5 below.

As was pointed out, (22) differs slightly from the result given by Graham & Kullar. Because of the complexity of the derivation, one would like some verification of the correctness of the present result. The fact that (21) and (35) are related by (30) gives some reassurance that there were no algebraic errors in the derivation, but this is not conclusive proof since (30) has not been independently verified.

For the two-dimensional incompressible case, the lift acts at the quarter-chord point as shown by von Kármán & Sears (1938). An interesting question asked by one of the reviewers is whether this also holds for the present analysis. For the incompressible case this interesting result derives from the fact that the distribution of loading behaves as $[(1-x)/(1+x)]^{\frac{1}{2}}$, independent of the gust frequency. This does not hold for $M \neq 0$ even for the first-order correction (Amiet 1974, equations (14)–(18)). Expressions for the pressure distribution are not given here since these generally are more complex and could not be calculated in closed form.

4.3. *Skewed gust in compressible flow*

As before, factoring $S(\lambda)$ from (25), the lift can be written in this limit as

$$\left[\frac{\beta C_L(\lambda,\nu,M)}{2\pi w_0 S(\lambda)}\right]_{\lambda=\alpha\nu\to 0} = 1+\mathrm{i}\lambda g+\tfrac{1}{2}g(\eta^2\nu^2-2\lambda^2 g)+\lambda^2\left(\ln\frac{\lambda}{\nu}+\frac{\mathrm{i}\pi}{2}+\ln 2\right)(C_0+\ln\lambda)$$
$$-\mathrm{i}\nu\lambda\eta\delta\left(C_0+\ln\lambda+\frac{\nu^2}{2\lambda^2}\right)+\tfrac{1}{2}\nu^2(C_0+\ln\lambda-\tfrac{1}{2})+\frac{\nu_1^2}{2\mathrm{i}\lambda}-\frac{\nu_1^4}{4\lambda^2}+O(\nu^3), \quad (38)$$

with an argument ν/λ assumed for g, η and δ. Expanding for $\nu \to 0$ with λ fixed,

$$\left\{\left[\frac{\beta C_L(\lambda,\nu,M)}{2\pi w_0 S(\lambda)}\right]_{\lambda=\alpha\nu\to 0}\right\}_{\nu\to 0} = 1+\frac{\mathrm{i}\lambda}{2}\left(\frac{\nu}{\lambda}\right)^2(C_0+\ln\lambda+2\ln 2-\gamma-\ln\nu+\tfrac{1}{2})-\frac{\nu^2}{8}+\frac{\nu_1^2}{2\mathrm{i}\lambda}$$
$$-(2\ln 2-\gamma-\ln\nu)\tfrac{1}{2}\nu^2(C_0+\ln\lambda+\tfrac{1}{2})-\tfrac{1}{2}\nu^2(C_0+\ln\lambda)^2+O(\nu^3), \quad (39)$$

which is the same as the $\lambda = O(\nu) \to 0$ limit of (24). Combining (24), (38) and (39) as in (31), the composite solution is found to be

$$\frac{\beta[C_L(\lambda,\nu,M)]_{composite}}{2\pi w_0 S(\lambda)} = 1+\mathrm{i}\lambda g\left\{(\gamma-2\ln 2+\ln\nu)\left[1+\frac{C(\lambda)}{\mathrm{i}\lambda}\right]+\frac{C(\lambda)-1}{\lambda^2}-\frac{1}{2}\right\}\frac{\nu^2}{2}-\frac{\mathrm{i}\nu_1^2}{2\lambda}-\frac{\nu_1^4}{4\lambda^2}$$
$$-\lambda^2 g^2-\lambda^2(C_0+\ln\lambda-\tfrac{1}{2})g+\frac{\nu^2}{2}\left(C_0+\ln\lambda+\frac{1}{\mathrm{i}\lambda}\right)(C_0+\ln\lambda-\gamma+2\ln 2-\ln\nu), \quad (40)$$

with an argument of ν/λ assumed for g so that

$$g(\nu/\lambda) = \ln(\nu/\lambda)-\tfrac{1}{2}\mathrm{i}\pi+\mathrm{i}\nu\eta\delta/\lambda-\ln 2 \quad (41)$$

and η and δ, also functions of ν/λ, given by (26).

Just as for the incompressible skewed gust case, this composite solution does not have the correct slope as $\lambda = 0$. A correction for this is given in §6 below.

5. Comparison with an exact solution

It should be noted that both ν_1 and ν_2 are required to be small for the general case of a skewed gust in compressible flow given by (24) and (25). If ν alone is small with $O(\nu_1) = O(\nu_2) = O(1)$, an approximate solution can be found by another method. In particular, an exact solution for the case $\nu = 0$ will be given.

The relevant differential equation for this problem is from equation (1) of Amiet (1974):

$$\left[\nabla^2 - M^2\left(\frac{b}{U}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2\right]\phi(x,y,z,t) = 0, \tag{42}$$

where the coordinates are normalized by the semichord b and y denotes the spanwise direction. Letting

$$\phi(x,y,z,t) = \phi_0(x,z)\,\mathrm{e}^{\mathrm{i}(\omega t+\lambda M^2 x+k_y y)} \tag{43}$$

(42) becomes

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\beta^2\partial z^2}-\nu_2^2+\nu_1^2\right]\phi_0 = 0. \tag{44}$$

Letting $\nu_1 = \nu_2$ and $z = Z/\beta$, this becomes just Laplace's equation, and the lift is given by equation (14) of Amiet (1974) or by Osborne (1973) as

$$\left[\frac{\beta C_L(\lambda,\nu,M)}{2\pi w_0 S(\lambda)}\right]_{\nu_1=\nu_2} = \left[J_0\left(\frac{\nu_1^2}{\lambda}\right)-\mathrm{i}J_1\left(\frac{\nu_1^2}{\lambda}\right)\right]. \tag{45}$$

Equation (45) is exact for $\nu = 0$, but there has been no expansion in ν for small but non-zero ν. It is worth noting that the solution of (2) for $\nu = 0$ is independent of the value of ν_1. The parameter ν_1 enters only through (6). The solution of (2) when $\nu_1 = \nu_2 \neq 0$ is the same solution as for the classic Sears case for which $\nu_1 = \nu_2 = 0$.

Now, a more accurate form of the composite solution is found by manipulating the terms to match the result given by (45) while not violating the other assumptions of the expansion. Thus, for the general case of a skewed gust in compressible flow (40) becomes

$$\begin{aligned}\frac{\beta C_L(\lambda,\nu,M)_{composite}}{2\pi w_0 S(\lambda)} &= \mathrm{e}^{\mathrm{i}\lambda g}\left[J_0\left(\frac{\nu_1^2}{\lambda}\right)-\mathrm{i}J_1\left(\frac{\nu_1^2}{\lambda}\right)\right] \\ &+\left\{(\gamma-2\ln 2+\ln\nu)\left[1+\frac{C(\lambda)}{\mathrm{i}\lambda}\right]+\frac{C(\lambda)-1}{\lambda^2}-\frac{1}{2}\right\}\tfrac{1}{2}\nu^2-\tfrac{1}{2}\nu_1^2 g-\tfrac{1}{2}\lambda^2 g^2 \\ &-(C_0+\ln\lambda-\tfrac{1}{2})\lambda^2 g+\tfrac{1}{2}\nu^2\left(C_0+\ln\lambda-\frac{\mathrm{i}}{\lambda}\right)(C_0+\ln\lambda-\gamma+2\ln 2-\ln\nu),\end{aligned} \tag{46}$$

with an argument of ν/λ assumed for g. A similar modification can be made to (32) and (37) for a skewed gust in compressible flow and a parallel gust in compressible flow, but the results are contained in (46). Equation (46) should also have a correction to make the slope zero for $\lambda \to 0$ as discussed below.

Of course, there is an arbitrariness in the form of the final result since it is not exact. One may try to cast the result in a form that satisfies some additional criterion. The first term on the right-hand side of (46) is the first-order solution of Amiet (1974, 1976*b*). At that time it was put in this form with the exponential factor, which is not required by (45), partly because this gives the proper ν^{-1} decay for large ν; the slope of the asymptote is then correct, even though the actual magnitude is not. Although the solution was not intended to be used for large ν, it was hoped that having the correct asymptotic slope would delay any significant deviation of the approximate solution from the exact solution.

6. A correction for small λ

Equation (33) shows that the variation with λ of the magnitude of the lift should have a zero slope for $\lambda \to 0$. However, (32) has an infinite slope at this point even though it is correct to $O(\nu_2^2)$ in the two limits $\nu_2 \to 0$ and $\lambda = O(\nu_2) \to 0$. For example, a simple expression which is zero to $O(\nu_2^2)$ under these two limits but has infinite slope at $\lambda = 0$ is

$$f_1(\lambda, \nu_2) = \lambda \nu_2^2 [\ln (\lambda^2 + \nu_2^2)^{\frac{1}{2}} - \ln \lambda]. \tag{47}$$

In fact, a term such as this can be used to cancel the infinite slope at $\lambda = 0$ without affecting the order of accuracy of the solution. If (29) is subtracted from (8) one finds the expression that, when added to (27), gives the composite solution, (32). That is, since the second term of (31) behaves properly at $\lambda = 0$, the difference between the first and third terms on the right-hand side of (31) contains the terms producing the infinite slope at $\lambda = 0$. An examination of this difference shows that the following result, which is very similar to (47), corrects the improper slope at $\lambda = 0$:

$$\begin{aligned}\frac{\text{Correction}}{2\pi w_0 \mathrm{i}\lambda \nu_2^2 S(\lambda)} &= (\gamma - 2\ln 2 + \ln \nu_2 + \tfrac{1}{2})\tfrac{1}{2}\{[C_0 + \ln (\lambda^2 + \nu_2^2)^{\frac{1}{2}}]^2 - (C_0 + \ln \lambda)^2\} \\ &\quad - \tfrac{1}{3}[C_0 + \ln (\lambda^2 + \nu_2^2)^{\frac{1}{2}}]^3 + \tfrac{1}{3}(C_0 + \ln \lambda)^3 - \tfrac{1}{4}[C_0 + \ln (\lambda^2 + \nu_2^2)^{\frac{1}{2}}] + \tfrac{1}{4}(C_0 + \ln \lambda).\end{aligned} \tag{48}$$

While maintaining the same order of approximation this can be simplified to

$$\frac{\text{Correction}}{\pi w_0 \mathrm{i}\lambda \nu_2^2 S(\lambda) \ln (1 + \nu_2^2/\lambda^2)^{\frac{1}{2}}} = (-\tfrac{1}{2}\mathrm{i}\pi - \ln 2 + \tfrac{1}{2})(2C_0 + \ln \nu_2 + \ln \lambda) - (C_0 + \ln \lambda)^2 - \tfrac{1}{2}. \tag{49}$$

There is another method of correcting the solution which the author finds preferable since it does not require adding additional terms to the equation. On carefully examining the origin of the terms producing the improper behaviour, one finds that changing the argument of $C(\lambda)$ and a few associated terms in (32) also corrects the improper behaviour. Thus, for the case of a skewed gust in incompressible flow the following equation gives the proper behaviour at $\lambda = 0$ while retaining the same order of accuracy:

$$\begin{aligned}\frac{[C_L(k_x, k_y)]_{composite}}{2\pi w_0 S(\lambda)} &= \mathrm{e}^{\mathrm{i}\lambda g} + \left\{(\gamma - 2\ln 2 + \ln \nu_2)\left[1 + \frac{C(\Theta)}{\mathrm{i}\Theta}\right] + \frac{C(\Theta) - 1}{\Theta^2} - \frac{1}{2}\right\}\tfrac{1}{2}\nu_2^2 - \tfrac{1}{2}\lambda^2 g^2 \\ &\quad - \lambda^2 g (C_0 + \ln \lambda - \tfrac{1}{2}) + \tfrac{1}{2}\nu_2^2\left(C_0 + \ln \Theta - \frac{\mathrm{i}}{\Theta}\right)(C_0 + \ln \Theta - \gamma + 2\ln 2 - \ln \nu_2),\end{aligned} \tag{50}$$

with $\Theta \equiv (\lambda^2 + \nu_2^2/e)^{\frac{1}{2}}$, $\lambda = k_x$ and $\nu_2 = k_y$. The factor e^{-1} in the definition of Θ is rather arbitrary since it was introduced solely to give better agreement with Graham's numerical solution at the point $\lambda = 0$. This is the only parameter in the solution that might be considered an 'adjustable constant'. Since the factor ν_2^2 in Θ is itself somewhat arbitrary, multiplying it by a factor other than unity is not such a great additional assumption.

Equation (50) further illustrates the intent of this paper. This is not simply to derive an accurate approximation to the lift function, but rather to further understand the solution and to cast it in as simple a form as possible while retaining the proper behaviour for all the appropriate limiting cases. Thus, the author finds the correction in (50) preferable to that in (49) since it strives to find a more proper structure for the equation rather than just adding a correction, even if this involves a certain amount of speculation in guessing the structure.

For the case of a skewed gust in compressible flow a similar modification of (46) gives

$$\frac{\beta C_L(\lambda,\nu,M)_{composite}}{2\pi w_0 S(\lambda)} = e^{i\lambda g}\left[J_0\left(\frac{\nu_1^2}{\lambda}\right) - iJ_1\left(\frac{\nu_1^2}{\lambda}\right)\right]$$
$$+\left\{(\gamma - 2\ln 2 + \ln\nu)\left[1+\frac{C(\Theta)}{i\Theta}\right] + \frac{C(\Theta)-1}{\Theta^2} - \frac{1}{2}\right\}\tfrac{1}{2}\nu^2 - \tfrac{1}{2}\nu_1^2 g - \tfrac{1}{2}\lambda^2 g^2$$
$$-\lambda^2 g(C_0+\ln\lambda-\tfrac{1}{2}) + \tfrac{1}{2}\nu^2(C_0+\ln\Theta - i/\Theta)(C_0+\ln\Theta-\gamma+2\ln 2-\ln\nu), \quad (51)$$

with an argument of ν/λ assumed for g. This is the most general result and will be used in the calculations below. The definition of Θ for (51) is the same as for (50). Note that there is no correction for a parallel gust, with (51) giving the same result as (46) when $\nu_2 = 0$.

7. Calculated results

In addition to giving a firmer theoretical grasp of the problem, the above results are useful for numerical calculations. To judge the accuracy of the results, they will be compared with the numerical results of Graham (1970*a*, *b*). The numerical results are the actual values of Graham, which the author wishes to thank Professor Graham for supplying. When $\nu_2 = 0$ the parameter ν corresponds to the parameter μ used by Amiet (1975) in which the (M, λ) variable space is divided into low-frequency and high-frequency regimes. In the same manner, the present solution will be used in conjunction with the same high-frequency solution to see how thoroughly the two solutions cover the λ, ν_1, ν_2 variable space. All figures presented here were drawn by a computer driven plotter, connecting calculated points using straight line segments. This eliminates the possibility of human error in the plotting process.

7.1. *Parallel gust in compressible flow*

First the $\nu_2 = 0$ case of a parallel gust in compressible flow is considered. Stated another way, this is the classical Sears problem, but with the addition of compressibility. From one viewpoint this is the simplest case since a correction near $\lambda = 0$, as given by (51), is not needed. Figure 2 shows several of the solutions discussed here. Two of the lines are the basic solutions of Graham & Kullar, one for $\nu_1 \to 0$ with fixed λ given by (21), the other for $\nu_1 = O(\lambda) \to 0$ given by (22). Also shown is the composite solution given by (37) and the improved composite solutions given by (46) or (51). The composite result is seen to be an improvement over the basic Graham & Kullar result when compared to the numerical results of Graham, and the improved composite is even better. The improvement obtained by combining the solutions into the composite form is better than one might have expected beforehand.

The comparison for the $\nu_2 = 0$ case is continued in figure 3 in which the results for several Mach numbers are compared with numerical results and with a high-frequency approximation. For this comparison the results are seen to be exceptionally good.

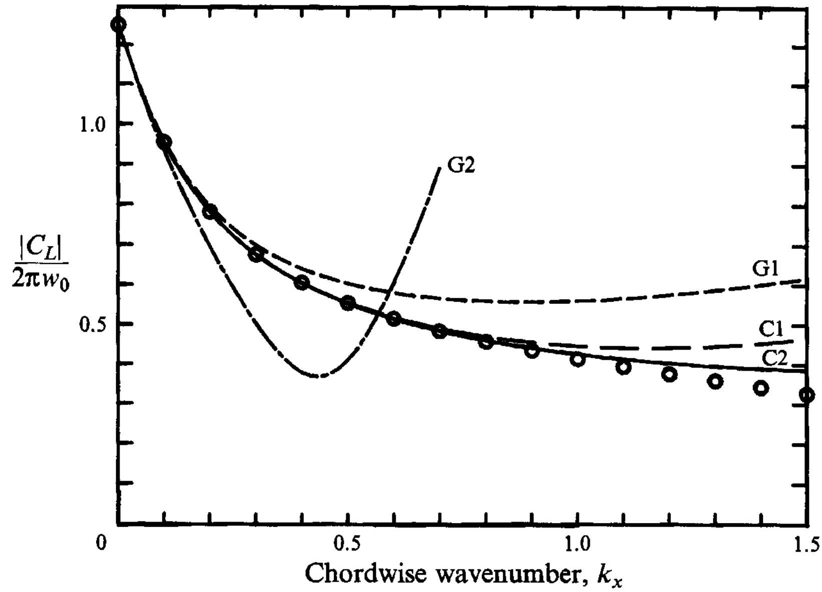

FIGURE 2. Comparison between the lift solutions for the parallel gust compressible flow case. $M = 0.6$. G1, Graham & Kullar first solution, (21); G2, Graham & Kullar's second solution, (22); C1, composite solution, (37); C2, improved composite, (46); symbols: Graham (1970*b*) numerical.

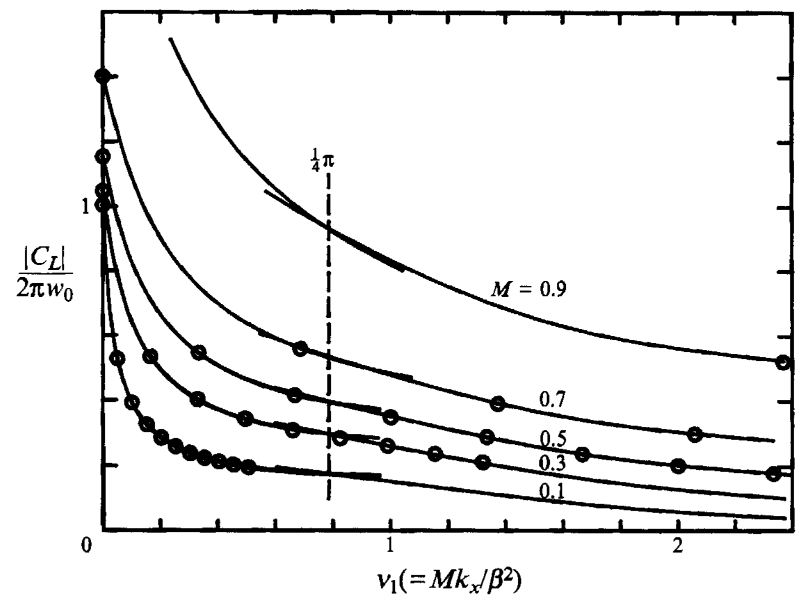

FIGURE 3. Small ν ($\nu_1 < \frac{1}{4}\pi$, (46)) and large ν ($\nu_1 > \frac{1}{4}\pi$, Amiet 1976*a*) lift solutions compared with the numerical solution (symbols) of Graham (1970*b*). $k_y = 0$.

Previously, the author has used either a criterion of $\nu_1 = 0.4$ or $\nu_1 = \frac{1}{4}\pi$ as the division between the low- and high-frequency regimes, depending on the particular approximate solutions used. With this improved low-frequency solution it appears that the value $\nu_1 = \frac{1}{4}\pi$ is a good value for the division for the parallel gust case. This accuracy does not appear to hold for the skewed gust case, $\nu_2 \neq 0$, as will be shown below. The parallel compressible gust case is a less stringent test of the solution procedure than is the case

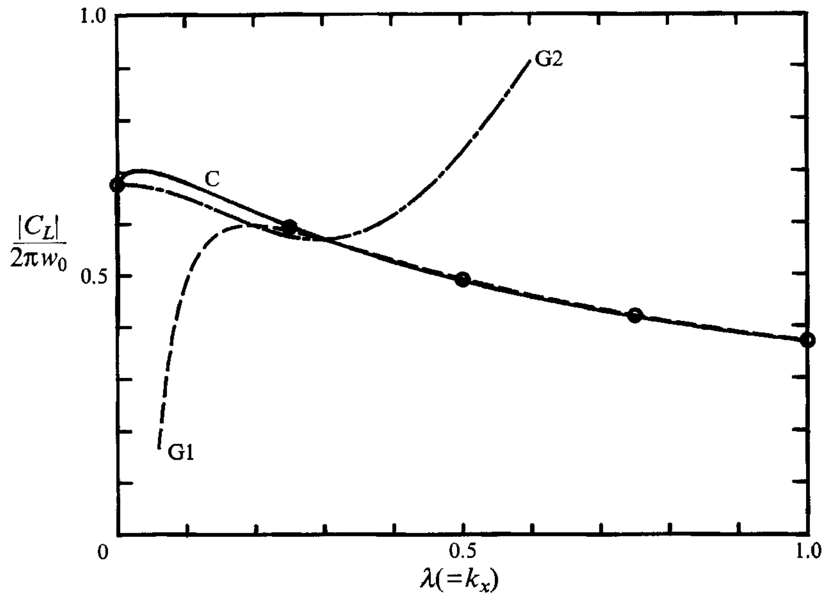

FIGURE 4. Comparison between the lift solutions for the skewed gust compressible flow case. $\nu = k_y = 0.25$. G1, Graham & Kullar first solution, (8); G2, Graham & Kullar second solution, (15); C, composite solution, (46); symbols: Graham (1970*a*) numerical.

of $M = 0$ with $\nu_2 \neq 0$. For the parallel compressible gust case $\nu \to i\nu_1 = iM\lambda$; but M is limited to $M < 1$ so for this case the range of ν is limited for a given value of λ. In other words, $\nu \to 0$ if $\lambda \to 0$ for a parallel compressible gust, but this is not necessarily the case for a skewed gust. As noted by Amiet (1975), this criterion of $\nu_1 = \frac{1}{4}\pi$ can be interpreted in terms of an average wavelength. If λ_1 is the acoustic wavelength for upstream travelling waves and λ_2 is the wavelength for downstream travelling waves, then setting $2\lambda_{avg}^{-1} = \lambda_1^{-1} + \lambda_2^{-1}$ leads to the result that $4\lambda_{avg} = c$. That is, compressibility begins to be important when the acoustic wavelength is smaller than four times the scattering body dimension.

7.2. *Skewed gust in incompressible flow*

As noted above, this case gives a more rigorous test for the approximate result than the parallel gust case since λ can be zero without ν being zero. Figure 4 shows a calculation for the case $k_y = 0.25$ and $M = 0$. Both results of Graham & Kullar, (8) and (15), are shown, along with the composite solution, (46), without the correction given by (49) or (51). Whereas the small-ν solution of Graham & Kullar for the $k_y = 0$ case in figure 2 gives no difficulty when $\lambda \to 0$, for the $k_y \neq 0$ case figure 4 shows that the small-ν solution, (8), becomes infinite when $\lambda \to 0$; in the first case $\nu_1 \to 0$ when $\lambda \to 0$, but in the second case ν_2 is fixed. Thus, the second form of Graham & Kullar, (15), is essential when $k_y \neq 0$. The composite solution gives an improvement over either solution alone, but the improper slope for $\lambda \to 0$ is clearly evident. The hump in the curve near $\lambda = 0$ is only of magnitude ν_2^2, as required by the approximations used, but it is a characteristic that is undesirable. The result corrected for this $\lambda \to 0$ difficulty using (50) is shown in figure 5. The slope at $\lambda = 0$ has been corrected to zero, completely eliminating the inappropriate hump in the curve in this region. Unfortunately, there are insufficient exact numerical points to give a thorough evaluation for the corrected result. It should be noted in figure 5 that (51) is not exact when $\lambda = 0$; it may appear so from the plot, but (51) is still only accurate to $O(\nu_2^2)$ for $M = \lambda = 0$.

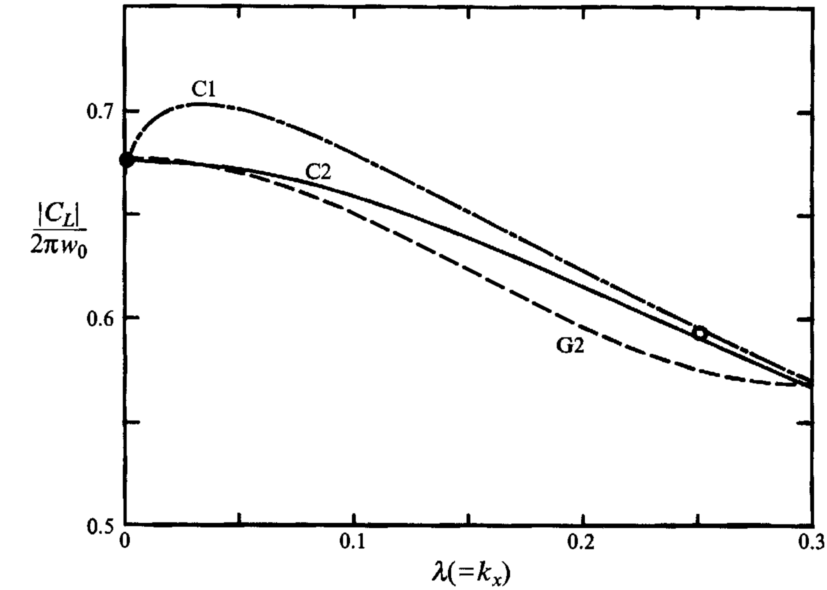

FIGURE 5. Effect of the small-λ correction on the composite solution for the skewed incompressible gust case. $\nu = k_y = 0.25$. G2, Graham & Kullar second solution (15); C1, composite solution, (46); C2, corrected composite solution, (50); symbols: Graham (1970*a*) numerical solution.

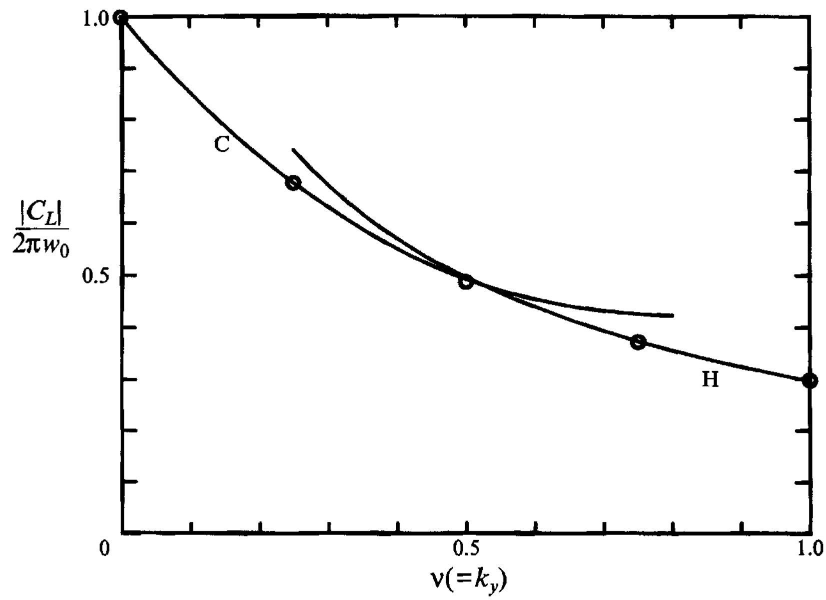

FIGURE 6. Small-ν and large-ν solutions compared with the numerical solution (symbols) of Graham (1970*a*). $k_x = M = 0$. C, Composite solution, (51); H, large-ν solution (Adamczyk 1974; Amiet 1976).

Finally, figure 6 shows the variation of the lift coefficient with ν for fixed λ, with both the present small-ν solution and the large-ν solution (Adamczyk 1974; Amiet 1976*a*) shown. For the parallel compressible gust case in figure 3 the value $\nu = \frac{1}{4}\pi$ was seen to be a good value at which to change from the small- to the large-ν solution. For the case of a skewed incompressible gust in figure 6 the small-ν solution is less accurate, and 0.5 appears to be a better value at which to change to the high-frequency solution.

8. Conclusions

For the parallel gust in compressible flow the composite solution, (51), gives results that are more accurate when compared to exact numerical results than might have been expected. This is partly due to the process of forming the composite solution which seems to cancel errors in the fundamental solutions that make up the composite solution and partly due to the fact that (51) was cast in a form that becomes an exact solution to the case of a flat-plate airfoil encountering a skewed gust in compressible flow with $\nu = 0$ ($\nu_1 = \nu_2$). This condition occurs both when $k_y = M = 0$ (the Sears problem) and when the sweep speed of the gust–airfoil intersection point, for a skewed gust, is sonic relative to the fluid ($\beta k_y = M k_x$). Mathematically, the composite solution is accurate to $O(\nu^2)$ for $\nu \to 0$ with fixed λ and for $\lambda = O(\nu) \to 0$. For the case of an incompressible skewed gust the accuracy is acceptable, but not as good as for the case of a parallel gust in compressible flow. The initial form of the composite solution gave the incorrect slope for $\lambda \to 0$ with fixed ν, but this was corrected either by adding a correction term given by (49) or by modifying the argument of $C(\lambda)$ as in (51). When used together with a solution for large ν, a good approximation to the lift can be made over the entire (ν_1, ν_2, λ) variable space for subsonic flow.

This paper is dedicated to Professor William R. Sears on the occasion of his 80th birthday.

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